Chapter 4

Number Theory

“Throughout the study of numbers, students ... should identify classes of numbers and examine their properties. For example, integers that are divisible by 2 are called even numbers and numbers that are produced by multiplying a number by itself are called square numbers. Students should realize that different types of numbers have particular characteristics; for example, square numbers have an odd number of factors and prime numbers have only two factors.”

—Principles and Standards for School Mathematics

“Number theory offers many rich opportunities for explorations that are interesting, enjoyable, and useful. These explorations have payoffs in problem solving, in understanding and developing other mathematical concepts, in illustrating the beauty of mathematics and in understanding the human aspects of the historical development of number.”

—Curriculum and Evaluation Standards for School Mathematics

Number theory is primarily concerned with the study of the properties of the natural numbers. Number theory topics, such as multiples, factors, prime numbers, prime factorization, least common multiples, and greatest common divisors, are an integral part of the elementary and middle school curricula. These concepts are used extensively when working with rational numbers.

In this chapter, you will use a variety of formats—concrete models, games, a grid-coloring activity, and geometric applications—to explore topics in number theory. The concrete models, coloring activity, and the geometric applications develop visual images of the number theory concepts involved. These multiple representations promote greater conceptual understanding by connecting the abstract ideas and the physical models that are used.
Activity 1: Great Divide Game

PURPOSE
Apply and reinforce divisibility rules.

MATERIALS
Other: Great Divide Game Board (page 57), a number cube, and chips for markers

GROUPING
Work in groups of two to four players.

GETTING STARTED
Follow the rules below to play the Great Divide Game.

RULES FOR THE GREAT DIVIDE GAME

1. Make a number cube labeled as follows. 2, 3, 4, 5, 9

2. To begin the game, each player rolls the number cube. The player with the greatest number goes first; play progresses to the next player on the right.

3. On each turn, the player rolls the number cube and places a chip on a number on the game board that is divisible by the number showing on the number cube.

   Example:
   Player rolls 3
   Player covers 168
   Score 1 point

   If the player can name other numbers on the number cube that are factors of the number that was covered, the player scores one point for each.

   Player names 2 and 4
   Score 2 points
   Total: 3 points for that turn

4. If the player is unable to find an uncovered number on the game board that is divisible by the number showing on the number cube, he or she must pass the number cube to the next player. If another player knows a play that can be made with the number on the number cube, that player may call attention to the mistake and tell the other players what uncovered number on the board is divisible by the number on the cube. The player citing the mistake may then place a chip on that number and earn points. This does not affect the turn of the player citing the mistake. If more than one player calls attention to a mistake, the first player to do so makes the play.

5. Players keep a running total of their scores. A player who cannot cover a number in three successive turns is eliminated from the game. When the game board is filled, or if all players have failed to play in three successive turns, the game ends. The player with the highest score is the winner.
Great Divide

Game Board

<table>
<thead>
<tr>
<th>168</th>
<th>435</th>
<th>105</th>
<th>702</th>
<th>180</th>
<th>156</th>
</tr>
</thead>
<tbody>
<tr>
<td>444</td>
<td>342</td>
<td>270</td>
<td>189</td>
<td>387</td>
<td>258</td>
</tr>
<tr>
<td>231</td>
<td>390</td>
<td>459</td>
<td>260</td>
<td>324</td>
<td>465</td>
</tr>
<tr>
<td>280</td>
<td>153</td>
<td>388</td>
<td>396</td>
<td>477</td>
<td>294</td>
</tr>
<tr>
<td>152</td>
<td>138</td>
<td>110</td>
<td>594</td>
<td>154</td>
<td>315</td>
</tr>
<tr>
<td>279</td>
<td>666</td>
<td>515</td>
<td>340</td>
<td>195</td>
<td>378</td>
</tr>
</tbody>
</table>

Divisibility Rules

2  A number is divisible by 2 if it is an even number.
3  A number is divisible by 3 if the sum of the digits is divisible by 3.
4  A number is divisible by 4 if the two-digit number formed by the tens and ones digits is divisible by 4.
5  A number is divisible by 5 if the ones digit is 0 or 5.
9  A number is divisible by 9 if the sum of the digits is divisible by 9.
Activity 2: A Square Experiment

PURPOSE
Develop the concepts of prime, composite, and square numbers using a geometric model.

MATERIALS
Pouch: Colored Squares
Online: Centimeter Graph Paper

GROUPING
Work individually or in groups of 3 or 4.

GETTING STARTED
Use squares or graph paper to form all the rectangular arrays possible with each different number of squares. Record your results in Table 1.

Examples:

<table>
<thead>
<tr>
<th>Number of Squares</th>
<th>Rectangular Arrays</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3, 3, 1</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6, 1, 3, 2, 3, 2</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Note: is not a rectangular array.

When an array is described by its dimensions, the figure has an altitude of 1 unit and a base of 2 units and is labeled $1 \times 2$. The figure is labeled $2 \times 1$.

TABLE 1

<table>
<thead>
<tr>
<th>Number of Squares</th>
<th>Dimensions of the Rectangular Arrays</th>
<th>Total Number of Arrays</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1, 3, 3, 1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6, 1, 3, 2, 3, 2, 3</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. Use the results from Table 1 to complete Table 2.

TABLE 2

<table>
<thead>
<tr>
<th>Number of Squares That Produced:</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only One Array</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Only Two Arrays</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>More Than Two Arrays</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>An Odd Number of Arrays</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Suppose you have 24 squares.
   
a. How many rectangular arrays can be made?
   
b. In which column(s) in Table 2 would you place 24?
   
c. What are the factors of 24?

3. a. What are the factors of 16?
   
b. How many rectangular arrays can you make with 16 squares?
   
c. In which column(s) of Table 2 would you place 16?

4. Look at the data in Table 1 and Table 2. How is the number of factors of a given number related to the number of rectangular arrays?

5. a. Why is it that the numbers in column D of Table 2 produce an odd number of arrays?
   
b. What are the next three numbers that would be placed in column D?
6. What is the mathematical name for the numbers in
   a. column B?
   b. column C?
   c. column D?

7. Which numbers can be placed in two lists? Why?

8. Can any numbers be placed in three lists? If so, which ones?

9. Write each of the following composite numbers as a product of primes.
   a. 28
   b. 42
   c. 150
   d. 231

10. a. Can every composite number be written as a product of primes? Explain your reasoning.
    b. If two people write the same number as a product of primes,
       i. how would their factorizations be alike?
       ii. how might the factorizations be different?
Activity 3: The Factor Game

PURPOSE
Develop the idea of the prime factorization of a number using a game.

MATERIALS
Pouch: 15 each, two different Colored Squares
Other: Six paper clips and the Factor Game Game Board (page 62)

GROUPING
Work in pairs or in teams of two students each.

GETTING STARTED
Play the Factor Game three times with your partner.

• To begin, one player places two paper clips on numbers in the factor list. The paper clips may be placed on the same number or on different numbers. The player then multiplies the numbers and places a square of his or her color on the product on the game board.

• Players then alternate turns. On a turn, a player may form a new product in one of the following ways:
  A. Place a new paper clip on any number in the factor list.
  B. Take one paper clip off a number in the factor list.
  C. Move one of the paper clips already on the factor list to a different number.

• A turn ends when a player places a colored square on a product, or the product is already covered or it is not on the game board.

• The first player to cover three adjacent products in a row horizontally, vertically, or diagonally with squares of his or her color is the winner.

When you have finished playing the game, answer the following questions.

1. Are the numbers on the game board prime or composite? the numbers in the factor list?

2. a. List all of the factors of 60. Identify which factors are prime, which are composite, and which are neither prime nor composite.

   b. List all the different ways that paper clips can be placed on numbers in the factor list to result in a product of 60.
## Factor Game

**Game Board**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>12</td>
<td>25</td>
<td>90</td>
<td>36</td>
</tr>
<tr>
<td>108</td>
<td>75</td>
<td>120</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>80</td>
<td>8</td>
<td>9</td>
<td>45</td>
</tr>
<tr>
<td>54</td>
<td>30</td>
<td>100</td>
<td>18</td>
<td>50</td>
</tr>
<tr>
<td>20</td>
<td>24</td>
<td>27</td>
<td>40</td>
<td>60</td>
</tr>
</tbody>
</table>

**Factor List**

2  
3  
5
Every composite number can be written as the product of prime factors. Such a product is the **prime factorization** of the number.

A **factor tree** can be useful for finding the prime factorization of a number. An example of a factor tree for 45 is given at the right.

1. Describe what happens in Step 1 of the factor tree for 45. Why is the 5 circled but not the 9?

2. Describe what happens in Step 2. Why do the branches stop at the circled numbers?

3. What is the prime factorization of 45?

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1. Complete the factor tree for 120 at the right. Why are more branches necessary to make this tree than to make the one for 45?

2. What is the prime factorization of 120?

3. Sketch a factor tree of your own for 120 that starts with a different pair of factors.

4. Compare the circled numbers at the ends of the branches in the two factor trees for 120. What do you notice?
Activity 4: A Sieve of Another Sort

PURPOSE
Investigate primes, composites, multiples, and prime factorizations using a “sieve.”

MATERIALS
Other: Orange, red, blue, green, and yellow colored pencils or crayons

GROUPING
Work individually.

GETTING STARTED
Eratosthenes, a Greek mathematician, invented the “sieve” method for finding primes over 2200 years ago. This activity explores a variation of Eratosthenes’ sieve.

As you discovered in Activity 2, one is neither prime nor composite. To show this, mark an X through 1.

The first prime number is 2. Color the diamond in which 2 is located orange. Use red to color the upper-left corner of the Key and the upper-left corner of all squares containing multiples of 2. Any number greater than 2 with a corner colored will fall through the sieve.

What was the first multiple of 2 that fell through the sieve? ______
The next uncolored number is 3. Color the diamond surrounding the 3 orange. Use blue to color the upper-right corner of the Key and the upper-right corner of all squares containing multiples of 3.

What was the first multiple of 3 that fell through the sieve? _____

Repeat this process for 5 and 7. Color the diamonds surrounding the numbers orange. Use green to color the lower-right corners of the Key and of all squares containing multiples of 5. Use yellow to color the lower-left corners of the Key and of all the squares containing multiples of 7. Note the first multiples of 5 and of 7 that fall through the sieve.

Finally, use orange to color the diamond surrounding all the numbers in the grid that are in squares with no corners colored. These numbers are all primes.

1. How do you know that 2, 3, 5, and 7 are prime numbers?

2. How can you tell that 2, 3, 5, and 7 are prime numbers from the way the sieve is colored?

3. How can you identify composite numbers from the way the sieve is colored?

When you colored the multiples of 2, the number 4 was the first multiple of 2 that fell through the sieve.

When you colored the multiples of 3, the number 9 was the first multiple of 3 that fell through the sieve.

1. When you colored multiples, what was
   a. the first multiple of 5 that fell through the sieve?
   b. the first multiple of 7 that fell through the sieve?

After you colored the multiples of 7, the next uncolored number was 11.

2. If you could color multiples of 11, what would be the first number to fall through the sieve?

3. When you color multiples of a prime number, how is the first multiple of the prime that falls through the sieve related to the prime number?

4. If the grid went to 300, what is the largest prime whose multiples must be colored before you can be certain that all of the remaining uncolored numbers are prime?

5. What is the largest prime less than 1000? Explain how you obtained your answer.
The sieve can be used for more than finding primes.

1. List the numbers that are colored with the code for 2 and for 3.

2. The numbers in Exercise 1 are multiples of both 2 and 3. What numbers are they?

3. How could you use the color code to find
   a. the multiples of 14?
   b. the multiples of 30?

The sieve can also help you find the prime factorization of a number.

**Example:** From the sieve, you find that the prime factors of 72 are 2 and 3.

\[ 72 = 2 \times 3 \times 12 \quad \text{(12 is not prime)} \]

Since 2 is a prime, you are done.

\[ 72 = 2^3 \times 3^2 \]

1. Find the prime factorization of
   a. 54
   b. 84
   c. 100

1. Pairs of prime numbers like 3 and 5 that differ by 2 are called *twin primes*. List all the twin primes less than 100.

2. What is the longest string of consecutive composite numbers on the grid?

3. Several of the numbers on the grid are divisible by three different primes. What is the smallest number that is divisible by four different primes?
Activity 5: Tiling with Squares

PURPOSE
Use a geometric model to explore the concept of the greatest common divisor (GCD) of two numbers and use the model to develop an algorithm for finding the GCD.

MATERIALS
Online: Half-centimeter Graph Paper

GROUPING
Work individually or in pairs.

GETTING STARTED
Tiling a region means to completely cover it with non-overlapping shapes. The study of tilings can lead to some interesting questions. For example:

If \( m \) and \( n \) are whole numbers, what is the length of a side of the largest square that can be used to tile an \( m \times n \) rectangle?

To answer this question, let’s look at some rectangles. For each problem, draw the rectangle on graph paper and either draw or cut out squares to tile it.

1. In Parts a–d, if a \( 4 \times 8 \) rectangle can be tiled with squares of the given size, make a sketch to show the tiling. If it can’t, explain why not.
   a. \( 1 \times 1 \)   b. \( 2 \times 2 \)   c. \( 3 \times 3 \)   d. \( 4 \times 4 \)

2. Can any other size square be used to tile a \( 4 \times 8 \) rectangle? Explain.

3. The results for a \( 4 \times 8 \) rectangle are summarized in the table. Fill in the data for the other rectangles listed in the table.

<table>
<thead>
<tr>
<th>Dimensions of Rectangle width ( \times ) length</th>
<th>Prime Factorization width ( \times ) length</th>
<th>Common Prime Factors</th>
<th>Dimensions of Squares That May Be Used</th>
<th>Length of Side of Largest Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \times 8 )</td>
<td>( 2 \times 2 )</td>
<td>( 2 \times 2 )</td>
<td>( 1 \times 1, 2 \times 2, 4 \times 4 )</td>
<td>( 4 )</td>
</tr>
<tr>
<td>( 6 \times 9 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 18 \times 30 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 24 \times 36 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 8 \times 15 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. How does the length of a side of the largest square that can be used to tile a rectangle appear to be related to the common prime factors of the width and length of the rectangle?
1. a. List the divisors (factors) of 18.
   b. List the divisors of 30.
   c. List the common divisors of 18 and 30.
   d. What is the GCD of 18 and 30?

2. a. List the divisors of 24.
   b. List the divisors of 36.
   c. List the common divisors of 24 and 36.
   d. What is the GCD of 24 and 36?

3. How do the greatest common divisors found in Exercises 1 and 2 compare to the Length of the Side of the Largest Square that can be used to tile the corresponding rectangle in the table?

4. If $m$ and $n$ are whole numbers, what is the length of a side of the largest square that can be used to tile an $m \times n$ rectangle?

1. a. List the divisors of 54, 72, and 90.
   b. List the common divisors of 54, 72, and 90.
   c. What is the GCD of 54, 72, and 90?

2. a. Find the prime factorizations of 54, 72, and 90.
   b. List the common prime factors of 54, 72, and 90.
   c. How does the GCD of 54, 72, and 90 appear to be related to the common prime factors of the numbers?

3. a. List the divisors of 84, 105, 126, and 210.
   b. List the common divisors of 84, 105, 126, and 210.
   c. What is the GCD of 84, 105, 126, and 210?

4. a. Find the prime factorizations of 84, 105, 126, and 210.
   b. List the common prime factors of 84, 105, 126, and 210.
   c. How is the GCD of 84, 105, 126, and 210 related to the common prime factors of the numbers?

5. Explain how the prime factorizations of a set of numbers can be used to find the greatest common divisor of the numbers.
Activity 6: Pool Factors

PURPOSE
Apply the concepts of greatest common divisor, least common multiple, and relatively prime numbers in a geometric problem situation.

MATERIALS
Online: Centimeter Graph Paper
Other: Straightedge

GROUPING
Work individually or in small groups.

GETTING STARTED
On a piece of graph paper, draw several pool tables like the one shown below, but with different dimensions. Label the pockets A, B, C, and D in order, starting with the lower-left pocket as shown.

Place a ball on the dot in front of pocket A. Shoot the ball as indicated by the arrows. The ball always travels on the diagonals of the grid and rebounds at an angle of 45 degrees when it hits a cushion.

Count the number of squares through which the ball travels.

Count the number of hits, that is, the number of times the ball hits a cushion, the initial hit at the dot, and the hit as the ball goes into a pocket.

In the table, enter the dimensions of each pool table, the number of squares through which the ball travels, and the number of hits. Analyze the data in the table and determine a rule that predicts the number of squares and the number of hits, given the dimensions of any pool table.

EXTENSIONS
Add a column headed **Final Pocket** to the table. In this column, for each pool table, record the letter of the pocket into which the ball finally fell. Use this data to find a rule that will predict which pocket the ball will fall into for any pool table.
Chapter Summary

The Great Divide Game in Activity 1 provided an opportunity for you to reinforce your skills in applying divisibility rules.

In Activities 2 and 4, you discovered that natural numbers can be classified by how many factors they have. Prime numbers have exactly two factors; composite numbers have more than two; square numbers have an odd number of factors; and one, which is in a class of its own, has exactly one factor.

Initially, this classification of the natural numbers may have seemed rather arbitrary. However, in Activities 2 and 3, you discovered that every natural number greater than one is either a prime number or can be expressed as a product of prime numbers, and that this product is unique except for the order of the factors. Thus, the prime numbers are the building blocks from which all natural numbers greater than one are constructed. This result is known as the Fundamental Theorem of Arithmetic.

Activities 5 and 6 were devoted to the study of factors and multiples of numbers. The greatest common divisor (GCD), also known as the greatest common factor (GCF), and procedures for calculating it were developed using a geometric model. Least common multiples (LCM) and greatest common divisors were applied in Activity 6 and will be used extensively in your study of the rational numbers.